Unstable periodic orbits and discretization cycles

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Limit cycles that arise from discretizing the variable(s) of a nonlinear map are generally found to shadow individual unstable periodic orbits (UPOs) of the corresponding continuous map. In a few cases the discretization cycles can only be explained with other mechanisms, such as the near-occurrence of an UPO, or crossover between two or more UPOs.

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I. INTRODUCTION

Discretized maps—also known as granular, integer or finite-state maps, iterations or machines—have been widely studied in the physics, mathematics, and biology literature [1–18]. Here discretization refers to the state variables, which are changed from real numbers to integers or binary rationals; not to time discretization. The physicist's interest in this problem arises from the need to understand the effects of iterating chaotic maps and random number generators with digital computers which store variables in finite registers, and which convert the usual continuous description into discrete mathematics.

In this paper we investigate the relation between discretization cycles (DCs), which arise from discrete-variable dynamics, and unstable periodic orbits (UPOs), which are found in continuous dynamics, and which help to determine the global behavior of dynamical systems.

In our study of the logistic map for five nonlinear parameter values and 9 degrees of discretization, most of the 130 DCs that we studied can be directly explained in terms of identified UPOs, or of several mechanisms that involve them.

Section II of this paper reviews integer maps, and defines several useful quantities. Section III describes the methods of the present study. Section IV presents the results, summarized in two Tables, one for DCs and the other for UPOs, along with a description of the correspondences found. Finally, Section V contains a discussion of the results.

II. INTEGER MAPS

We consider maps $x_{n+1}=f(x_n)$, where x is a vector of continuous variables in one or several dimensions. The long-time behavior of $f(x_n)$ can include, among others, limit cycles, chaotic attractors, or quasiperiodic behavior. For an integer map the phase space of vector x is broken into N segments, usually taken to be equally sized and spaced. We refer to N as the map's discretization number.

The continuous function $f(x_n)$ then is transformed into a map of an integer set, e.g., $[0,1,2,\ldots,N-1]$ onto itself, in which each integer (or vector) is used to label a segment or area of the granular phase space. The new map is obtained by calculating the image of the central point of each segment, and assigning it the integer label of the segment whose central point is either nearest, nearest from above, or nearest from below. These schemes are known as round-off, roundup, and truncation, respectively. The result is a deterministic integer map in which each discrete state has a unique discrete image. In this paper we consider roundoff, the most standard procedure.

The simultaneous presence of a finite number of states and of deterministic dynamics leads to the unavoidable existence of DCs, often with transient states leading to them. The ensuing phase space is a collection of directed graphs which embody the dynamics, and which are known as de Bruijn diagrams [2,4,5].

In this paper we are concerned with DCs, but not with transients. While most previous studies have concentrated on the approximate scaling laws for the irregular dependence of number of cycle and transient lengths and basin sizes as a function of N, in this paper we explore the connection between discrete cycles and the unstable periodic orbits that occur in the corresponding continuous map.

For this study we have chosen the logistic equation, $x_{n+1}=ax_n(1-x_n)$, with parameter (a) values of 3.6, 3.7, 3.8, 3.9, and 3.99, all of them chaotic, and which include band chaos $(a=3.6)$ as well as fully developed chaos $(a=3.99)$. We used discretization numbers in powers of 2, from 32 to 8192, corresponding to 5–13 bits of machine precision.

III. METHODS

Three methods were used to explore the correspondence between UPOs and DCs.

First, exhaustive enumeration and listing of all DCs for each pair of (a,N), obtained by iterating the discrete map from every possible initial condition and by systematically labeling the cycles with the smallest value of any state in the cycle.

Second, exhaustive enumeration of UPOs of length $m\leqslant 7$ in the continuous map, obtained by systematically finding the roots of $g_{a,m}(x)=f^{(m)}(a,x)-x$, where $f^{(m)}(a,x)$ is the mth iterate of the logistic map with parameter a. This required eliminating complex roots, as well as roots of iterates that are divisors of m which are automatically roots of $g_{a,m}(x)$.

Finally, *ad hoc* search for roots of $g_{a,m}(x)$ in the continu-

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ous map for $8 \leqslant m \leqslant 20$, with initial trial values of x equal to those of points on an identified DC, in order to attempt to find an UPO that is shadowed by the DC in question.

All these calculations were performed in MATHEMATICA [19]. The cutoff values $m=7$ and $m=20$ in the second and third procedures were dictated by CPU time and accuracy constraints in our computer. They seem to be sufficient, however, to allow a complete picture of the UPO-DC correspondence to emerge, as shown in the following section.

IV. RESULTS

Results are summarized in Table I for DCs and Table II for the continuous map's UPOs, respectively. For all combinations of a,N, 130 DCs were identified. For the continuous map, 106 UPOs were found, 67 of them as part of the exhaustive $N \leqslant 7$ enumeration. The obvious fixed point $x=0$ is not included in this count.

The convention in Table I is as follows. The approximately 100 DCs that are not followed by any symbol can be clearly associated with a continuous map UPO from Table II. The almost 20 DCs followed by (x) are of a high enough period ($m \geqslant 20$) that it has been prohibitive to identify a candidate UPO. The six DCs followed by (d) can be identified with UPOs of *half* the length (e.g., a period-2 DC and an unstable fixed point). In this case, an artifact of the discretization produces higher-order DCs than the continuous map's UPOs, exploiting derivatives near ± 1 of iterate of $f^{(m)}(a,x)$. We also note that some of the period-12 and 16 DCs and UPOs for $a=3.6$ seem to interweave the period 4 UPO for the same parameter value. Therefore, period length multiplication is not an effect uniquely associated with discretization.

The two remaining cases are the most intriguing. The five followed by (+) are fairly small DCs, with periods $5 \leqslant m \leqslant 8$, which ought to have corresponding UPOs. However, they do not. In all cases, we found that $g_{a,m}(x)$ *nearly* touches the x axis at the points that form the unexplained DC. For example, $g_{3.9,8}(x)$ shows a minimum of 0.095 at $x=0.0852$, consistent with the smallest state of the period-8 DC found for the same value of a and discretization number $N=128$. Similarly, $g_{3.99,5}(x)$ shows a minimum of 0.01 at $x=0.0097$, consistent with the smallest state of the period-5 DC found for $a=3.99$ and discretization number of $N=2048$. It appears that the roundoff process itself plays a role in turning these minima into roots in all five instances.

Second, the six cases followed by (*) are small enough ($12 \leqslant m \leqslant 18$) that a corresponding UPO should have been found by the third method of the preceding section. However, it was not. These occur for values of a (3.9, 3.99) with a high density of points on identified UPOs. The DCs can satisfactorily be explained by crossovers between small sections of shorter UPOs. For example, $m=15$, $a=3.99$, $N=256$ DC successively shadows pieces of period 7, 7, 15, and 7 UPOs. Similarly, the $m=18$, $a=3.9$, $N=512$ DC shadows four distinct UPOs of period 6, 7, 6, and 7, respectively.

We expect that most, if not all of the longer unaccounted-for cycles followed by (x), can be eventually explained by one of the mechanisms described above. A single $N=21$

TABLE I. Discrete limit cycles (DC) for diverse combinations of parameter value a and discretization number N for the logistic map. Symbols: (x), no unstable periodic orbit (UPO) match found so far; (+), near-roots; (d), doubling of period of corresponding UPO; (*), shadowing of several UPOs; no symbol, corresponding UPO found.

a	N	DC cycle lengths
3.6	32	1, 2, 6(+)
	64	4
	128	2(d), 12
	256	1, 16
	512	4, 8
	1024	2(d), 4, 4(d), 30(x)
	2048	1, 2, 4, 12, 14, 16
	4096	2, 4, 12, 14, 32(x)
	8192	2, 2(d), 124(x)
3.7	32	6
	64	11
	128	23(x)
	256	6, 9
	512	9, 10
	1024	36(x)
	2048	2(d), 6, 14, 16, 31(x)
	4096	1, 19
	8192	1, 2, 6, 8, 9, 21
3.8	32	4, 5
	64	1, 2, 5, 6
	128	5, 11
	256	24(x)
	512	5, 6, 7
	1024	2(d), 14
	2048	1, 38(x)
	4096	1, 2, 39(x)
	8192	2, 4, 21(x), 32(x), 63(x)
3.9	32	3
	64	3, 5
	128	2, 6, 8(+)
	256	3, 5(+), 15
	512	3, 8, 18(*)
	1024	2, 21(x)
	2048	1, 2, 5, 18
	4096	2, 8, 10, 23(x)
	8192	2(d), 142(x)
3.99	32	1, 2, 3
	64	1, 2, 9
	128	1, 3, 5
	256	1, 5, 15(*)
	512	12(*), 18(*)
	1024	7, 13(*), 20(x)
	2048	2, 4, 5(+), 6, 9, 12, 15(*), 36(x)
	4096	5(+), 7, 24(x)
	8192	1, 4, 6, 7, 114(x)

TABLE II. Identified unstable periodic orbits (UPOs). For each parameter value a, the top line shows the UPOs found in the systematic enumeration of periods of length $N \leq 7$. The second line shows higher-order UPOs found in the specific search for matches to observed DCs. Parentheses indicate more than one UPO of a given length.

a	UPO lengths
3.6	1, 2, 4
3.6	8, 12(2), 14(2), 16(2)
3.7	1, 2, 4, 6(2)
3.7	8, 9(2), 10, 11, 14, 16, 19, 21
3.8	1, 2, 4, 5(2), 6(2), 7(4)
3.8	11, 14
3.9	1, 2, 3(2), 4, 5(2), 6(3), 7(6)
3.9	8(3), 10, 15, 18
3.99	1, 2, 3(2), 4(3), 5(4), 6(7), 7(14)
3.99	9(2), 12(3), 13(2), 15(4), 18(3), 20

cycle with $a=3.7$ and $N=4192$ was positively tested with procedure 2 from the preceding section.

V. DISCUSSION

In this paper we have begun to address the origin of limit cycles in discrete dynamical systems. We find that, for cycle lengths within our computational power, the shadowing of individual unstable periodic orbits in the corresponding continuous map explains most of the observed limit cycles. We have discovered several additional mechanisms for the genesis of discrete cycles: period multiplication, the stabilization of near-roots of the function $g_{a,m}(x)$ defined in Sec. III, and the piecewise shadowing of several UPOs of different lengths.

The extension of this work to continuous-time dynamical systems is far from a straightforward matter, as the identification of UPOs in that case is very laborious [20]. It may be worthwhile to attempt this extension in order to confirm experimental findings [21] of limit cycles in inherently discrete insect populations, in cases where such cycles have been reported not to follow unstable periodic orbits, but where the near-root mechanism may apply.

The greatest mystery in this topic still remains open: why are some UPOs shadowed rather than others? The selection mechanism probably involves the complex interplay between the dynamics and the discretization scheme. For example, the errors introduced after the N time steps in the cycle must bring the system close enough to the original value of the continuous value for the DC to be generated.

Finally, we offer the conjecture that an analysis of the difference series between the continuous and discrete systems, equivalent to the measurement noise series introduced in Ref. [22], may provide a way to address this selection issue.

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